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ON INDEPENDENT PARTITIONS OF VERTEX SET IN A GRAPH

K. K. Dechamma, K. R. Rajanna*, R. Rajendra and P. Siva Kota Reddy*****

Department of Mathematics,
T. John Institute of Technology,
Bangalore - 560083, Karnataka, INDIA

E-mail : dechdeepi@gmail.com

*Department of Mathematics,
Acharya Institute of Technology, Bangalore - 560107, Karnataka, INDIA

E-mail : rajanna@acharya.ac.in

**Department of Mathematics,
Field Marshal K. M. Cariappa College Madikeri - 571201, INDIA

E-mail : rrajendrar@gmail.com

***Department of Mathematics,
JSS Science and Technology University,
Mysuru - 570006, Karnataka, INDIA

E-mail : pskreddy@jssstuniv.in

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Abstract: We define independent partition of the vertex set $V(G)$ as a partition of $V(G)$ into independent sets in a graph G. The concepts of independent graph of a graph G with respect to an independent partition and a dummy independent set in an independent partition have been introduced. Some results on these concepts are presented.

Keywords and Phrases: Independent Set, Dummy Independent Set, Independent Partition, Independent Graph.

2020 Mathematics Subject Classification: 05Cxx, 05C69.

1. Introduction

For basic terminologies we follow the text-book of Harary [2] and Deo [1]. The cardinality of a set S is denoted by $|S|$. A partition of a set S is a collection of disjoint non-empty subsets of S whose union is S. The number of partitions of a finite set S with n elements is the n-th Bell's number and is denoted by B_n (see [3]).

Let $G = (V, E)$ be a graph (finite, undirected, simple). The degree of a vertex v in G is denoted by $\deg(v)$. We denote the maximum degree among all the vertices of G by Δ and minimum degree among all the vertices of G by δ. If two vertices u and v are adjacent in G, we write $u \sim v$; otherwise we write $u \not\sim v$.

Two vertices u and v in a graph G are said to be fused if the two vertices are replaced by a single new vertex (ab) such that every edge that was incident on either u or v or on both is incident on the new vertex. If all the vertices in a subset U of $V(G)$ are fused, then the corresponding single new vertex is denoted by $\langle U \rangle$.

A subset I of $V(G)$ is an independent set (or stable set) in G if no two vertices in I are adjacent in G. A maximal independent set in G is an independent set that is not a subset of any other independent set. The parameter $\alpha_0(G) = \max\{|I| : I \text{ is an independent set in } G\}$ is called the independence number of G. Any independent set I with $|I| = \alpha_0(G)$ is called a maximum independent set.

The chromatic number $\chi(G)$ of a graph G is the minimal number of colours needed to colour the vertices in such a way that no two adjacent vertices have the same colour.

When a planar graph is drawn with no crossing edges, it divides the plane into a set of regions, called faces. Each face is bounded by a closed walk called the boundary of the face. By convention, we also count the unbounded area outside the whole graph as one face. The degree of the face is the length of its boundary.

In section 1, we discuss independent partition of the vertex set $V(G)$ that is a partition of $V(G)$ into independent sets in a graph $G = (V, E)$. The concept of independent graph of a graph G with respect to an independent partition is discussed in section 2. In section 3, the concept of a dummy independent set in an independent partition has been introduced and some results are presented. All graphs assumed in this paper are finite, simple and undirected.

2. Independent Partitions

Definition 2.1. *Let $G = (V, E)$ be a graph. An independent partition of V is a partition of V into independent sets in G.*

A collection $P = \{V_1, \ldots, V_k\}$ of some subsets of V is an independent partition of V (or an independent partition in G) if

(i) V_i is an independent set for each i,

(ii) $V_i \cap V_j = \emptyset$, for $i \neq j$,

(iii) $\cup_{i=1}^{k} V_i = V$.

Given a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the collection $P = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is an independent partition of V. This independent partition in G is called the discrete independent partition. In the complete graph K_n, there is only one independent partition that is the discrete independent partition, because the only independent sets in K_n are singletons.

Definition 2.2. *An independent partition P of a graph G is finest if there is no independent partition P' of G such that $|P'| < |P|$.*

The following proposition is immediate from the Definition 2.1:

Proposition 2.3. *If G is a graph with an independent partition P, then G is a $|P|$-partite graph.*

Corollary 2.4. *Let G be a graph with n vertices and P be any independent partition of $V(G)$. If $|P| = p$, then for each k, $0 \leq k \leq n-p$, there exist an independent partition P' of G with $|P'| = p + k$.*

Proof. If $|P| = p$, then G is a p-partite graph and it is easy to see that G is a $(p+k)$-partite graph for $0 \leq k \leq n-p$. So, there is an independent partition P' of G with $|P'| = p + k$ for each k, $0 \leq k \leq n-p$.

Proposition 2.5. *For any graph G, there exists a finest independent partition of V.*

Proof. Let G be a graph. Let k be the chromatic number of G. Then there is a proper colouring of G with k colours, say $c_1, c_2, ..., c_k$. Let V_i be the set of all vertices with colour c_i in this proper colouring, $1 \leq i \leq k$. Then $P = \{V_1, \ldots, V_k\}$ is a finest independent partition of V.

Second Proof of Proposition 2.5. Let G be a graph. Consider the set

$$S = \{|P| : P \text{ is an independent partitions of } G\}.$$

The set S is a non-empty set of positive integers, because the singleton vertex sets form an independent partition of V. By the well-ordering principle S contains a least positive integer l. So, there exist an independent partition P of G such that $|P| = l$. Clearly, P is a finest independent partition of V.

The number l in the above proof is nothing but chromatic number of G.

Remark 2.6. *For a graph G, a finest independent partition of V need not be unique. For instance, consider the graph G given in Figure 1. The chromatic number of G is 3. $P = \{\{v_1\}, \{v_2\}, \{v_3, v_4\}\}$ and $P = \{\{v_1, v_3\}, \{v_2\}, \{v_4\}\}$ are two different finest independent partitions of G.*

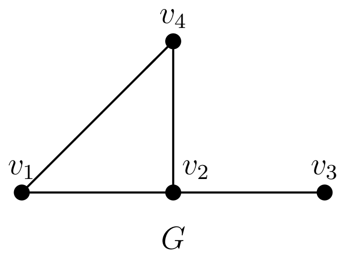


Figure 1: A graph with two finest independent partitions

The following propositions are immediate from the Definition 2.2:

Proposition 2.7. *Any two finest independent partitions of a graph G have same cardinality.*

Proposition 2.8. *Let G be a graph and P be a finest independent partition of V. If P' is an independent partition of V with $|P'| = |P|$, then P' is a finest independent partition of V.*

Proposition 2.9. *Let G be a graph. Then*

(i) $\chi(G) \leq |P|$, *for any independent partition P of G.*

(ii) $\chi(G) = |P|$, *for any finest independent partition P of G.*

The following corollaries are immediate from Proposition 2.9:

Corollary 2.10. *Let G be a graph with $n \geq 4$ vertices and P be any finite independent partition of V. Then G is 4-colourable if and only if $|P| \leq 4$.*

Corollary 2.11. *If P is any finite independent partition of a planar graph G with $n \geq 5$ vertices, then $|P| \leq 5$.*

Proof. Since every planar graph is 5-colourable, the proof follows from Proposition 2.9.

Theorem 2.12. *If P is an independent partition in a graph G , then there is a finest independent partition P' with $|P'| \leq |P|$.*

Proof. Let G be a graph. Suppose that P is an independent partition of a graph G. Consider the set

$$S = \{|Q| : Q \text{ is an independent partition of } V \text{ with } |Q| \leq |P|\}.$$

Note that $|P| \in S$ and hence the set S is a non-empty set of positive integers. By the well-ordering principle S contains a least positive integer l. So, there exist an independent partition P' of G such that $|P'| = l$. Clearly, P' is a finest independent partition of V and $|P'| \leq |P|$.

Theorem 2.13. *Let $p(G)$ be the number of independent partitions in a connected graph G with n vertices. Then*

(i) $p(G) = 1$, *if* $G \cong K_n$;

(ii) $2 \leq p(G) \leq B_n - 1$, *if* $G \ncong K_n$;

(iii) $p(G) = 2$, *if* $G \cong K_n - e$, *the graph obtained by deleting an edge in* K_n.

Proof.

(i) Suppose that $G \cong K_n$. Then the independent sets in G are singleton sets and hence there is only one independent partition in G, namely, the discrete independent partition. Consequently, $p(G) = 1$.

(ii) Suppose that $G \ncong K_n$. Then we can find two non-adjacent vertices, say, u, v in G. Apart from the discrete independent partition, the partition $\{\{u, v\}\} \cup \{\{x\} \mid x \in V - \{u, v\}\}$ of V is an independent partition in G. Therefore, there are at least two independent partitions in G. Therefore,

$$p(G) \geq 2 \tag{1}$$

Since $|V| = n$, there are B_n number of partitions of V. Since G is connected, V is not an independent set and hence $\{V\}$ is not an independent partition in G. Therefore

$$p(G) \leq B_n - 1 \tag{2}$$

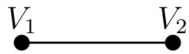
Now, from (1) and (2), we have $2 \leq p(G) \leq B_n - 1$.

(iii) Suppose that $G = K_n - e$. Then there are exactly two non-adjacent vertices, say, u, v in G. Hence there are exactly two independent partitions in G, namely, the discrete independent partition, the partition $\{\{u, v\}\} \cup \{\{x\} \mid x \in V - \{u, v\}\}$ of V. Therefore, $p(G) = 2$.

3. Independent Graph of a Graph with respect to an Independent Partition

Definition 3.1. *The independent graph $I_P(G)$ of a graph G with respect to an independent partition P of $V(G)$ is the graph with vertex $V(I_P(G)) = P$ and two vertices V_1 and V_2 are adjacent in $I_P(G)$ if there exist $u \in V_1$ and $v \in V_2$ such that $uv \in E(G)$.*

Example 3.2. Let $K_{m,n}$ be the complete bipartite graph with partite sets V_1 and V_2. Then $P = \{V_1, V_2\}$ is an independent partition (finest) of $V(K_{m,n})$ and the independent graph $I_P(K_{m,n})$ of $K_{m,n}$ w.r.to P is the following graph.



$I_P(K_{m,n})$

Remark 3.3. *From the Example 3.2, in view of $K_{3,3}$ it follows that, if an independent graph of a graph G is planar, then G need not be planar.*

Theorem 3.4. *Let G be a graph.*

(i) If P is an independent partition in G consisting of singleton sets i.e., P is the discrete independent partition, then $I_P(G) \cong G$.

(ii) If P is a finest independent partition of V with $|P| = n$, then $I_P(G) \cong K_n$.

(iii) If P_1 and P_2 are two finest independent partitions of G, then $I_{P_1}(G) \cong I_{P_2}(G)$.

Proof.

(i) Follows by Definition 3.1 as an isomorphism between $I_P(G)$ and G can be obtained by defining a map $V(I_P(G)) \to V(G)$ by $\langle\{v\}\rangle \mapsto v$.

(ii) Suppose that $P = \{V_1, V_2, \ldots, V_n\}$ is a finest independent partition of V with $|P| = n$. We claim that $V_i \sim V_j$ for all $i \neq j$ in $I_P(G)$. For if $V_i \nsim V_j$ for some $i \neq j$, without loss of generality we assume that $V_1 \nsim V_2$. Then no vertex in V_1 is adjacent to any vertex V_2. hence, it is clear that $V_1 \cup V_2$ is an independent set. Consider $\overline{P} = \{V_1 \cup V_2, V_3, \ldots, V_n\}$. We see that $\overline{P}$ is an independent partition of V with $|\overline{P}| = n - 1$, which contradicts to the fact that P is a finest independent partition of V. Hence our claim holds and consequently, $I_P(G) \cong K_n$.

(iii) If P_1 and P_2 are two finest independent partitions of G, then $|P_1| = |P_2|$ and hence from (ii), it follows that $I_{P_1}(G) \cong I_{P_2}(G) \cong K_n$, where $n = |P_1| = |P_2|$.

The following corollaries are immediate from Theorem 3.4:

Corollary 3.5. *If G is a bipartite graph with (finest) independent partition $P = \{V_1, V_2\}$ consisting of partite sets V_1 and V_2, then $I_P(G) \cong K_2$.*

Corollary 3.6. *If G and G' are isomorphic graphs with finest independent partitions P and P', respectively, then $I_P(G) \cong I_{P'}(G')$.*

Corollary 3.7. *Let G be a graph and P be a finest independent partition of G. We have* *(a) $|P| = 1$ if and only if G is a trivial graph.*

(b) $|P| = 2$ if and only if G is a bipartite graph.

(c) If $|P| = 3$, then there is a cycle of odd length in G.

(d) If there is a cycle of odd length in G, then $|P| \geq 3$.

Theorem 3.8. *Let G be a graph and $P = \{V_1, \ldots, V_k\}$ be an independent partition of V. The independent graph $I_P(G)$ is isomorphic to the graph obtained from G by fusing all the vertices in V_i, for each i and replacing parallel edges by a single edge, if there any.*

Proof. Let G_f be the graph obtained from G by fusing all the vertices in V_i, for each i and replacing parallel edges by a single edge, if there any. Define $\phi : V(I_P(G)) \to V(G_f)$ by $\phi(V_i) = \langle V_i \rangle$, $1 \leq i \leq k$. Clearly, ϕ is bijective and adjacency preserving map. Hence ϕ is an isomorphism and so $I_P(G) \cong G_f$.

Proposition 3.9. *Let G be a graph. Then*

(i) $\chi(G) \leq \chi(I_P(G))$, for any independent partition P of G.

(ii) $\chi(G) = \chi(I_P(G))$, for any finest independent partition P of G.

Proof. Follows from Proposition 2.9.

4. Dummy Independent Set in an Independent Partition

Definition 4.1. *Let G be a graph and $P = \{V_1, \ldots, V_k\}$ be an independent partition of V. An independent set $V_i \in P$ is called a dummy independent set (DIS) in P if for each $v \in V_i$, there exist some $V_j \in P$ with $j \neq i$ such that $V_j \cup \{v\}$ is an independent set in G.*

Note that an independent partition of a graph G may or may not have dummy independent sets.

Example 4.2. In the complete graph K_n on vertices, there is no dummy independent set in the discrete independent partition which is the only independent partition in K_n.

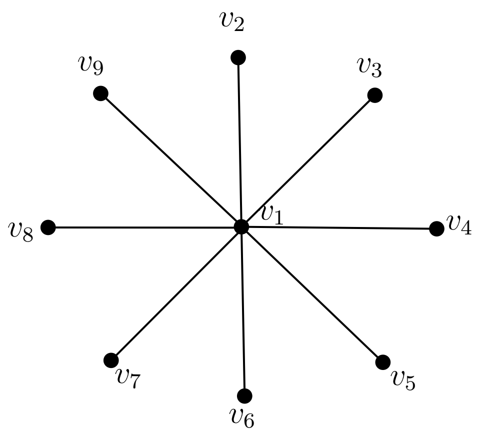
Example 4.3. If G is a graph $n > 2$ vertices that is not complete graph, then there is a vertex with v with $\deg(v) < n - 1$. It is easy to see that $\{v\}$ is a dummy independent set in the discrete independent partition in G.

Proposition 4.4. *If P is an independent partition of a graph G having only maximal independent sets, then there is no dummy independent set in P.*

Proof. Follows by Definition 4.1 and Definition of a maximal independent set in a graph.

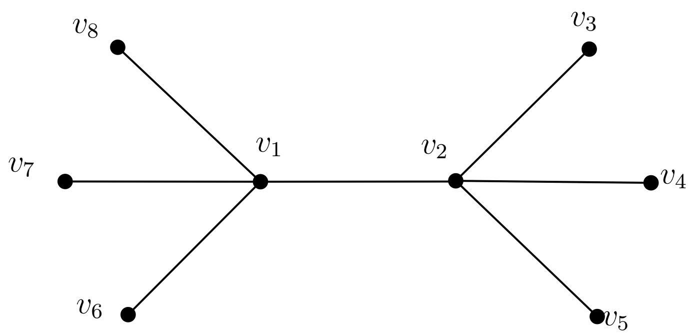
Remark 4.5. *A maximal (as well as a maximum) independent set in an independent partition of a graph G may or may not be a dummy independent set. Instances are provided in Examples 4.6 and 4.7.*

Example 4.6. Consider the star $K_{1,8}$ shown below. Let $P = \{V_1, V_2\}$, where $V_1 = \{v_1\}$ and $V_2 = \{v_2, \ldots, v_9\}$. The partition P is a finest independent partition in $K_{1,8}$. We see that V_1 and V_2 are maximal independent sets. Note that V_2 is a maximum independent set. Here V_1 and V_2 are not dummy independent sets in the independent partition P.



$K_{1,8}$

Example 4.7. Consider the double star graph G shown below. Let $P = \{V_1, V_2, V_3\}$, where $V_1 = \{v_1\}$, $V_2 = \{v_2\}$ and $V_3 = \{v_3, \ldots, v_8\}$. The partition P is an independent partition in G. We see that V_3 is a maximal (as well as maximum) independent set in G and it is a dummy independent set in P.



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Theorem 4.8. *If P is an independent partition of a graph G having a dummy independent set, then there is an independent partition P' with $|P'| < |P|$.*
Proof. Let G be a graph and $P = \{V_1, \ldots, V_k\}$ be an independent partition G. Suppose that there is a dummy independent set in P. Without loss of generality let us assume that V_1 is a dummy independent set in P. Let $V_1 = \{u_1, \ldots, u_m\}$. By Definition 4.1, for each u_r, there exist $V_j \in P$ with $V_j \neq V_1$ such that $V_j \cup \{u_r\}$ is an independent set in G. We define U_j, $2 \leq j \leq k$, by $U_j = V_j \cup S_j$, where $S_j = \{v \in V_1 \mid V_j \cup \{v\}$ is an independent set with j is the least index with this property$\}$. Then we have

(i) U_j is an independent set for each j,

(ii) $U_r \cap U_s = \emptyset$, for $r \neq s$,

(iii) $\cup_{j=2}^{k} U_j = V$.

Consider $P' = \{U_2, \ldots, U_k\}$. Then P' is an independent partition G with $|P'| = k - 1 < |P|$.

Corollary 4.9. *A finest independent partition of a graph G has no dummy independent set.*
Proof. Follows from Definition 2.2 and Theorem 4.8.

Corollary 4.10. *If P is an independent partition of a graph G having a dummy independent set, then there is a finest independent partition P' with $|P'| < |P|$.*
Proof. Let G be a graph. Consider the set

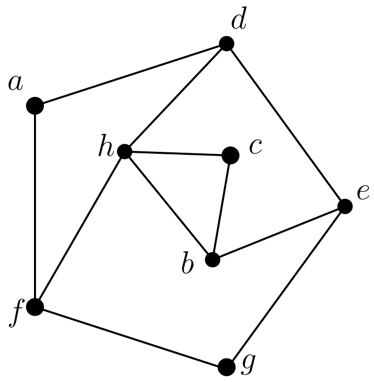
$$S = \{|Q| : Q \text{ is an independent partitions of } G \text{ with with } |Q| < |P|\}.$$

By Theorem 4.8, it follows that the set S is a non-empty set of positive integers. By the well-ordering principle S contains a least positive integer l. So, there exist an independent partition P' of G such that $|P'| = l$. Clearly, P' is a finest independent partition of V and $|P'| < |P|$.

Remark 4.11. *If P is an independent partition of a graph G with no dummy independent set, then P need not be a finest independent partition of V. For example, in the following graph G, the set $P = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}\}$ is an independent partition of a graph G with no dummy independent set, but it is not finest. The set $\{\{a, h\}, \{d, b, g\}, \{c, e, f\}\}$ is a finest independent partition of V.*

Theorem 4.12. *Let G be a graph and $P = \{V_1, \ldots, V_k\}$ be an independent partition of V. If $|P| \geq \Delta + 2$, then there is a dummy independent set in P.*
Proof. Suppose that $|P| = k \geq \Delta + 2$. Since $\deg(v) \leq \Delta$, $\forall v \in V(G)$, it follows

that, for each vertex $u \in V_1$ there exist at least one $V_j \in P$ with $j \neq 1$ such that $V_j \cup \{u\}$ is an independent set in G. Hence V_1 is a dummy independent set in P.



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Corollary 4.13. *If G is a planar graph with $\Delta \leq 3$, then $\chi(G) \leq 4$. In particular, every 3-regular planar graph has chromatic number ≤ 4.*

Proof. Suppose that G is a planar graph with $\Delta \leq 3$. If G has at most 4 vertices, then there is nothing to prove. We assume that G has at least 5 vertices. By Five-Color Theorem [1, Theorem 8-11, p.188-189], $\chi(G) \leq 5$. Hence there is an independent partition $P = \{V_1, \ldots, V_k\}$ of G with $|P| = k \leq 5$. We consider the following cases: (i) $|P| = k \leq 4$ and (ii) $|P| = k = 5$.

Case (i): If $|P| = k \leq 4$, clearly $\chi(G) \leq 4$.

Case (ii): If $|P| = k = 5$, by Theorem 4.12, there is a dummy independent set in P. Now, by the Corollary 4.10, there is a finest independent partition P' in G with $|P'| < |P| = 5$ i.e., $|P'| \leq 4$. Therefore $\chi(G) \leq 4$.

Acknowledgments

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